

## COMPLETE HOMOLOGY OVER ASSOCIATIVE RINGS

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ABSTRACT. We compare two generalizations of Tate homology to the realm of associative rings: stable homology and the J-completion of Tor, also known as complete homology. For finitely generated modules, we show that the two theories agree over Artin algebras and over commutative noetherian rings that are Gorenstein, or local and complete.

## INTRODUCTION

Tate introduced homology and cohomology theories over finite group algebras. In unpublished work from the 1980s—eventually given an exposition by Goichot [13]—P. Vogel generalized both theories to the realm of associative rings. These generalizations are now referred to as *stable (co)homology*. Tate’s theories achieved alternate generalizations through works of Triulzi and his adviser Mislin who introduced the *J-completion* of Tor [20] and the *P-completion* of covariant Ext [16]. An *I-completion* of contravariant Ext was studied by Nucinkis [17]. On the cohomological side, the P-completion of Ext agrees with stable cohomology; see Kropholler’s survey [15] and Appendix B.

In this paper, we investigate the homological side: Under what conditions does stable homology Tor agree with the J-completion Tor of Tor? Either theory enjoys properties that the other may not have: Complete homology has a universal property, and stable homology is a homological functor. The universal property of complete homology provides a comparison map from stable homology to complete homology. Our main theorem identifies conditions under which stable and complete homology agree. For some of these conditions, we do not know if they also ensure that the comparison map is an isomorphism; in fact, we ask in A.5 if the comparison map may, in some cases, fail to detect such agreement.

**Main Theorem.** *Let  $R$  be an Artin algebra, or a commutative Gorenstein ring, or a commutative complete local ring. For every finitely generated right  $R$ -module  $M$  and all  $i \in \mathbb{Z}$ , there are isomorphisms*

$$\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \widetilde{\mathrm{Tor}}_i^R(M, -)$$

*of functors on the category of finitely generated left  $R$ -modules.*

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By a commutative Gorenstein ring we mean a commutative noetherian ring whose localization at every prime ideal is a Gorenstein local ring. A commutative local ring is tacitly assumed to be noetherian. The isomorphisms in the Main Theorem hold for all modules over rings with finite *sfl* invariant; see Corollary 2.11. Among these rings are von Neumann regular rings and Iwanaga-Gorenstein rings; the latter are noetherian rings with finite injective dimension on either side. See Colby [7] and Emmanouil and Talelli [10] for details.

Complete homology and stable homology are defined for all modules over any associative ring, but we do not know if they agree in that generality. For a restricted class of modules, Iacob [14] has generalized Tate homology to the setting of associative rings. We do show that the necessary and sufficient conditions for stable homology to agree with Tate homology are also necessary and sufficient for complete homology to agree with Tate homology. However, complete and stable homology may agree even when Tate homology is not defined; see 3.7 and 3.8.

## 1. INJECTIVE COMPLETION OF COVARIANT FUNCTORS

In this paper, rings are assumed to be associative algebras over a fixed commutative ring  $\mathbb{k}$ . Let  $R$  be a ring; we adopt the convention that an  $R$ -module is a left  $R$ -module, and we refer to right  $R$ -modules as modules over the opposite ring  $R^\circ$ . The functors we consider in this paper are tacitly assumed to be functors from the category of  $R$ -modules to the category of  $\mathbb{k}$ -modules.

**1.1.** Let  $\mathcal{I}$  be a non-empty subset of  $\mathbb{Z}$ . If  $U = \{U_i \mid i \in \mathcal{I}\}$  and  $T = \{T_i \mid i \in \mathcal{I}\}$  are families of covariant functors, then by a *morphism*  $v: U \rightarrow T$  we mean a family  $\{v_i: U_i \rightarrow T_i \mid i \in \mathcal{I}\}$  of morphisms.

The next definition can be found in Triulzi's thesis [20, 6.1.1]. It is similar to the definitions of Mislin's of  $P$ -completion of a covariant cohomological functor [16, 2.1] and Nucinkis'  $I$ -completion of a contravariant cohomological functor [17, 2.4].

**1.2 Definition.** Let  $T = \{T_i \mid i \in \mathcal{I}\}$  be a family of covariant functors. The *J-completion* of  $T$  is a family  $\check{T} = \{\check{T}_i \mid i \in \mathcal{I}\}$  of covariant functors together with a morphism  $\tau: \check{T} \rightarrow T$  that have the following properties.

- (1) One has  $\check{T}_i(E) = 0$  for every injective  $R$ -module  $E$  and every  $i \in \mathcal{I}$ .
- (2) If  $U = \{U_i \mid i \in \mathcal{I}\}$  is a family of covariant functors with  $U_i(E) = 0$  for every injective  $R$ -module  $E$  and all  $i \in \mathcal{I}$ , and if  $v: U \rightarrow T$  is a morphism, then there exists a unique morphism  $\sigma: U \rightarrow \check{T}$  such that  $\tau\sigma = v$ .

It follows from part (2) above that the  $J$ -completion of  $T$  is, if it exists, unique up to unique isomorphism. To discuss existence we recall the notions of homological functors and satellites, the latter from Cartan and Eilenberg [3, chap. III].

**1.3.** A family  $T = \{T_i \mid i \in \mathbb{Z}\}$  of covariant additive functors is called a *homological functor* if for every exact sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  of  $R$ -modules, there is an exact sequence

$$\cdots \rightarrow T_i(N') \rightarrow T_i(N) \rightarrow T_i(N'') \xrightarrow{\delta_i} T_{i-1}(N') \rightarrow \cdots$$

of  $\mathbb{k}$ -modules with natural connecting homomorphisms  $\delta_i$ . For example, the family  $\text{Tor}^R(M, -) = \{\text{Tor}_i^R(M, -) \mid i \in \mathbb{Z}\}$  is such a functor for every  $R^\circ$ -module  $M$ .

**1.4.** Let  $N$  be an  $R$ -module and  $N \xrightarrow{\sim} I$  be an injective resolution. For every  $n \geq 0$  an  $n^{\text{th}}$  cosyzygy of  $N$  is  $\text{Ker}(I^n \rightarrow I^{n+1})$ ; up to isomorphism and injective summands, it is independent of the choice of injective resolution. If the resolution is minimal, then we use the notation  $\Omega^n N$  for the cosyzygy  $\text{Ker}(I^n \rightarrow I^{n+1})$ ; these modules are unique up to isomorphism. Notice the isomorphism  $\Omega^0 N \cong N$ .

Let  $T$  be a covariant functor. For  $n \geq 0$  the  $n^{\text{th}}$  right satellite of  $T$  is a functor, denoted  $S^n T$ , whose value at an  $R$ -module  $N$  is the cokernel of the homomorphism  $T(I^{n-1}) \rightarrow T(\Omega^n N)$ . Notice that one has  $S^0 T \cong T$  and  $S^{k+1} T(N) \cong S^1 T(\Omega^k N)$ .

**1.5.** Following Mislin's line of proof of [16, thm. 2.2], Triulzi [20, prop. 6.1.2] shows that every homological functor  $T = \{T_i \mid i \in \mathbb{Z}\}$  has a J-completion  $\tau: \check{T} \rightarrow T$  with  $\check{T}_i(N) = \lim_{k \geq 0} S^k T_{k+i}(N)$  for every  $R$ -module  $N$  and every  $i \in \mathbb{Z}$ .

**1.6 Remark.** Let  $T = \{T_i \mid i \in \mathbb{Z}\}$  be a homological functor. Fix  $i \in \mathbb{Z}$ ; every exact sequence  $0 \rightarrow \Omega^{k-1} N \rightarrow I^{k-1} \rightarrow \Omega^k N \rightarrow 0$  yields an exact sequence

$$T_{k+i}(I^{k-1}) \longrightarrow T_{k+i}(\Omega^k N) \xrightarrow{\delta_{k+i}} T_{k-1+i}(\Omega^{k-1} N).$$

The connecting homomorphism  $\delta_{k+i}$  induces a homomorphism from the cokernel  $S^k T_{k+i}(N)$  to  $T_{k-1+i}(\Omega^{k-1} N)$ . Composed with the natural projection onto  $S^{k-1} T_{k-1+i}(N)$  it yields a homomorphism  $S^k T_{k+i}(N) \rightarrow S^{k-1} T_{k-1+i}(N)$ . These homomorphisms provide the maps in the inverse system from 1.5.

The next result shows how to compute the J-completion directly from cosyzygies.

**1.7 Lemma.** Let  $T = \{T_i \mid i \in \mathbb{Z}\}$  be a homological functor and let  $N$  be an  $R$ -module. For every  $i \in \mathbb{Z}$  there is an isomorphism

$$\check{T}_i(N) \cong \lim_{k \geq 0} T_{k+i}(\Omega^k N).$$

*Proof.* Let  $N \xrightarrow{\sim} I$  be a minimal injective resolution of  $N$ . For each  $k \geq 1$  consider the exact sequence  $0 \rightarrow \Omega^{k-1} N \rightarrow I^{k-1} \rightarrow \Omega^k N \rightarrow 0$ . As  $T$  is a homological functor, there is an exact sequence

$$\cdots \rightarrow T_{k+i}(I^{k-1}) \longrightarrow T_{k+i}(\Omega^k N) \xrightarrow{\delta_{k+i}} T_{k+i-1}(\Omega^{k-1} N) \longrightarrow T_{k+i-1}(I^{k-1}) \rightarrow \cdots$$

Consider the commutative diagram

$$\begin{array}{ccc} T_{k+i}(\Omega^k N) & \xrightarrow{\delta_{k+i}} & T_{k+i-1}(\Omega^{k-1} N) \\ & \searrow & \nearrow \varphi^k \\ & S^k T_{k+i}(N) & \end{array}$$

The connecting homomorphisms  $\delta_{k+i}$  yield an inverse system. They also make up a morphism between copies of this inverse system, and its limit  $\lim_{k \geq 0} \delta_{k+i}$  is, by the universal property of limits, the identity on  $\lim_{k \geq 0} T_{k+i}(\Omega^k N)$ . It follows that

$$\lim_{k \geq 0} \varphi^k: \lim_{k \geq 0} S^k T_{k+i}(N) \longrightarrow \lim_{k \geq 0} T_{k+i}(\Omega^k N)$$

is surjective, and it is injective as  $\lim$  is left exact. Thus, one has

$$\check{T}_i(N) = \lim_{k \geq 0} S^k T_{k+i}(N) \cong \lim_{k \geq 0} T_{k+i}(\Omega^k N).$$

□

It is not given that the J-completion of a homological functor is itself a homological functor, but dimension shifting still works.

**1.8 Lemma.** *Let  $T = \{T_i \mid i \in \mathbb{Z}\}$  be a homological functor and let  $N$  be an  $R$ -module. For every  $n \geq 0$ , there is an isomorphism  $\check{T}_i(\Omega^n N) \cong \check{T}_{i-n}(N)$ .*

*Proof.* For  $n = 0$  the isomorphism is trivial, and by induction it is sufficient to prove it for  $n = 1$ . The isomorphisms in the next computation hold by Lemma 1.7:

$$\begin{aligned} \check{T}_i(\Omega N) &\cong \lim_{k \geq 0} T_{k+i}(\Omega^k(\Omega N)) \\ &= \lim_{k \geq 0} T_{(k+1)+(i-1)}(\Omega^{k+1} N) \\ &= \lim_{k \geq 1} T_{k+(i-1)}(\Omega^k N) \\ &\cong \check{T}_{i-1}(N). \end{aligned} \quad \square$$

**1.9 Lemma.** *Let  $v: \{U_i \mid i \in \mathbb{Z}\} \rightarrow \{T_i \mid i \in \mathbb{Z}\}$  be a morphism of homological functors such that  $v_i$  is an isomorphism for  $i \gg 0$ . If  $U_i(E) = 0$  holds for each injective  $R$ -module  $E$  and every  $i \in \mathbb{Z}$ , then the unique morphism that exists by Definition 1.2,  $\sigma: \{U_i \mid i \in \mathbb{Z}\} \rightarrow \{\check{T}_i \mid i \in \mathbb{Z}\}$ , is an isomorphism.*

*Proof.* Set  $\mathcal{I} = \{i \in \mathbb{Z} \mid v_i: U_i \rightarrow T_i \text{ is an isomorphism}\}$ . By assumption  $\mathcal{I}$  contains all integers sufficiently large. Since  $U$  vanishes on injective modules, it follows that  $\{U_i \mid i \in \mathcal{I}\}$  is the J-completion of  $\{T_i \mid i \in \mathcal{I}\}$ , i.e.,  $U_i \cong \check{T}_i$  for all  $i \in \mathcal{I}$ ; see Definition 1.2.

Now let  $j \in \mathbb{Z}$ , choose an  $i \in \mathcal{I}$  with  $i \geq j$ , and set  $n = i - j$ . For every  $R$ -module  $N$  there are isomorphisms

$$\begin{aligned} \check{T}_j(N) &= \check{T}_{i-n}(N) \cong \check{T}_i(\Omega^n N) \\ &\cong U_i(\Omega^n N) \\ &\cong U_{i-n}(N) = U_j(N). \end{aligned}$$

The first isomorphism holds by Lemma 1.8, and the second holds by the argument above as  $i$  is in  $\mathcal{I}$ . The last isomorphism follows by dimension shifting, as  $U$  is a homological functor that vanishes on injective modules.  $\square$

## 2. COMPARISON TO STABLE HOMOLOGY

We now focus on the J-completion of the homological functor  $\text{Tor}^R(M, -)$ .

**2.1 Definition.** Let  $M$  be an  $R^\circ$ -module; for the J-completion of the homological functor  $\text{Tor}^R(M, -) = \{\text{Tor}_i^R(M, -) \mid i \in \mathbb{Z}\}$ , see Definition 1.2, we use the notation  $\widetilde{\text{Tor}}^R(M, -) = \{\widetilde{\text{Tor}}_i^R(M, -) \mid i \in \mathbb{Z}\}$ . For every  $R$ -module  $N$ , the  $\mathbb{Z}$ -indexed family of  $\mathbb{k}$ -modules  $\widetilde{\text{Tor}}_i^R(M, N)$  is called the *complete homology* of  $M$  and  $N$  over  $R$ .

**2.2.** If  $M$  has finite flat dimension, or if  $N$  has finite injective dimension, then the complete homology  $\widetilde{\text{Tor}}^R(M, N)$  vanishes: this follows straight from the isomorphisms  $\widetilde{\text{Tor}}_i^R(M, N) \cong \lim_{k \geq 0} \text{Tor}_{k+i}^R(M, \Omega^k N)$  from Lemma 1.7.

The goal of this section is to compare complete homology to stable homology; we recall the definition of stable homology and refer to [4, sec. 2] or [13] for details.

**2.3.** Let  $M$  be an  $R^\circ$ -module and  $N$  be an  $R$ -module. Let  $P \xrightarrow{\sim} M$  be a projective resolution and  $N \xrightarrow{\sim} I$  be an injective resolution. The tensor product complex  $P \otimes_R I$  has components  $(P \otimes_R I)_n = \coprod_{i \in \mathbb{Z}} P_{i+n} \otimes_R I^i$ ; it is a subcomplex of  $P \overline{\otimes}_R I$  with components  $(P \overline{\otimes}_R I)_n = \prod_{i \in \mathbb{Z}} P_{i+n} \otimes_R I^i$  and differential extended by linearity from  $p \otimes i \mapsto \partial(p) \otimes i + (-1)^{|p|} p \otimes \partial(i)$ , where  $|p|$  is the (homological) degree of  $p$ . The quotient complex is denoted  $P \widetilde{\otimes}_R I$ , and the modules

$$\widetilde{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_{i+1}(P \widetilde{\otimes}_R I)$$

make up the *stable homology* of  $M$  and  $N$  over  $R$ . They fit in a long exact sequence with homology modules  $\mathrm{Tor}_i^R(M, N)$  and  $\overline{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(P \overline{\otimes}_R I)$ ; see [4, 2.5].

**Comparison via the universal property.** Let  $M$  be an  $R^\circ$ -module. Stable homology  $\widetilde{\mathrm{Tor}}^R(M, -)$  is a homological functor, and there is a morphism

$$\mathfrak{O}: \widetilde{\mathrm{Tor}}^R(M, -) \longrightarrow \mathrm{Tor}^R(M, -)$$

given by connecting maps in a long exact sequence; see [4, 2.5]. Stable homology  $\widetilde{\mathrm{Tor}}^R(M, E)$  vanishes for every injective  $R$ -module  $E$ ; see [4, 2.3]. By the universal property of complete homology, there is thus a morphism

$$\sigma: \widetilde{\mathrm{Tor}}^R(M, -) \longrightarrow \overline{\mathrm{Tor}}^R(M, -)$$

with  $\tau\sigma = \mathfrak{O}$  where  $\tau: \overline{\mathrm{Tor}}^R(M, -) \rightarrow \mathrm{Tor}^R(M, -)$  is as in Definition 1.2.

We aim for an explicit description of the morphisms  $\tau$ ,  $\mathfrak{O}$ , and  $\sigma$ . To that end we first reinterpret the computation of complete homology in Lemma 1.7.

**2.4.** Fix  $i \in \mathbb{Z}$ . Recall from the proof of Lemma 1.7 that the maps in the inverse system on the right-hand side of  $\widetilde{\mathrm{Tor}}_i^R(M, N) \cong \lim_{k \geq 0} \mathrm{Tor}_{k+i}^R(M, \Omega^k N)$  come from the connecting homomorphisms

$$(1) \quad \delta_{k+i}: \mathrm{Tor}_{k+i}^R(M, \Omega^k N) \longrightarrow \mathrm{Tor}_{k+i-1}^R(M, \Omega^{k-1} N).$$

Let  $P \xrightarrow{\sim} M$  be a projective resolution and let  $N \xrightarrow{\sim} I$  be a minimal injective resolution. There is an exact sequence  $0 \rightarrow I^{\geq k} \rightarrow I^{\geq k-1} \rightarrow \Sigma^{-(k-1)} I^{k-1} \rightarrow 0$ , for every  $k \geq 1$ , which induces an exact sequence

$$(2) \quad 0 \longrightarrow P \otimes_R \Sigma^k I^{\geq k} \longrightarrow P \otimes_R \Sigma^k I^{\geq k-1} \longrightarrow P \otimes_R \Sigma I^{k-1} \longrightarrow 0.$$

For every  $n \geq 0$  the canonical map  $\Omega^n N \rightarrow \Sigma^n I^{\geq n}$  is a minimal injective resolution. It follows that the exact sequence in homology associated to (2) yields homomorphisms from

$$\mathrm{H}_{k+i}(P \otimes_R \Sigma^k I^{\geq k}) = \mathrm{Tor}_{k+i}^R(M, \Omega^k N) \quad \text{to}$$

$$\mathrm{H}_{k+i}(P \otimes_R \Sigma^k I^{\geq k-1}) = \mathrm{H}_{k+i-1}(P \otimes_R \Sigma^{k-1} I^{\geq k-1}) = \mathrm{Tor}_{k+i-1}^R(M, \Omega^{k-1} N).$$

That these maps yield an inverse system isomorphic to the one given by the maps (1) is due to the diagram

$$\begin{array}{ccc} \mathrm{H}_{k+i}(P \otimes_R \Sigma^k I^{\geq k}) & \longrightarrow & \mathrm{H}_{k+i-1}(P \otimes_R \Sigma^{k-1} I^{\geq k-1}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Tor}_{k+i}^R(M, \Omega^k N) & \xrightarrow{\delta_{k+i}} & \mathrm{Tor}_{k+i-1}^R(M, \Omega^{k-1} N) \end{array}$$

where the top horizontal map is induced by the embedding  $I^{\geq k} \rightarrow I^{\geq k-1}$ . The diagram is commutative because  $P \otimes_R -$  is a triangulated functor on the derived

category, and  $H(-)$  is a homological functor on the derived category. This explains the second isomorphism in the chain

$$(3) \quad \begin{aligned} \widetilde{\mathrm{Tor}}_i^R(M, N) &\cong \lim_{k \geq 0} \mathrm{Tor}_{k+i}^R(M, \Omega^k N) \\ &\cong \lim_{k \geq 0} H_{k+i}(P \otimes_R \Sigma^k I^{\geq k}) \cong \lim_{k \geq 0} H_i(P \otimes_R I^{\geq k}) \end{aligned}$$

where the limits in the last line are taken over the system given by the maps

$$(4) \quad \iota^k: H_i(P \otimes_R I^{\geq k}) \longrightarrow H_i(P \otimes_R I^{\geq k-1})$$

induced by the embeddings  $I^{\geq k} \rightarrow I^{\geq k-1}$ .

**2.5.** Adopt the notation from 2.4. In view of 2.4(3), an element in complete homology  $\widetilde{\mathrm{Tor}}_i^R(M, N)$  is a sequence of homology classes  $([w^{\geq k}])_{k \geq 0}$  where each  $w^{\geq k}$  is a cycle in  $(P \otimes_R I^{\geq k})_i$ , and one has  $\iota^k([w^{\geq k}]) = [w^{\geq k-1}]$ . With this notation,

$$\tau_i: \widetilde{\mathrm{Tor}}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M, N) \quad \text{is given by} \quad ([w^{\geq k}])_{k \geq 0} \longmapsto [w^{\geq 0}];$$

cf. [20, proof of 1.2.2 and text after 6.1.2].

A homogeneous cycle  $z$  in  $P \otimes_R I$  of homological degree  $|z|$  is represented by a family  $(z^j)_{j \geq 0}$  with  $z^j \in P_{j+|z|} \otimes_R I^j$ , such that  $\partial(z)$  belongs to  $(P \otimes_R I)_{|z|-1}$ . That is,  $z$  may have infinite support and need not be a cycle in  $P \widetilde{\otimes}_R I$ , but  $\partial(z)$  has finite support, and by the definition of the differential,  $\partial(z)$  is a cycle in  $P \otimes_R I$ . Thus, an element in  $\widetilde{\mathrm{Tor}}_i^R(M, N)$  is the class  $[z]$  of a cycle in  $(P \widetilde{\otimes}_R I)_{i+1}$  with  $\partial(z)$  a cycle in  $(P \otimes_R I)_i$ , and with this notation the connecting homomorphism

$$\delta_i: \widetilde{\mathrm{Tor}}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M, N) \quad \text{is given by} \quad [z] \longmapsto [\partial(z)].$$

Let  $z = (z^j)_{j \geq 0}$  be a cycle in  $(P \widetilde{\otimes}_R I)_{i+1}$ ; for every  $k \geq 0$  set  $z^{\geq k} = (z^j)_{j \geq k}$ . We just saw that  $\partial(z)$  has finite support, so for every  $k \geq 0$  the boundary  $\partial(z^{\geq k})$  is a cycle in  $(P \otimes_R I^{\geq k})_i$ . In  $P \otimes_R I^{\geq k-1}$  the difference  $\partial(z^{\geq k-1}) - \partial(z^{\geq k}) = \partial(z^{k-1})$  is a boundary, so the sequence of homology classes  $([\partial(z^{\geq k})])_{k \geq 0}$  is compatible with the morphisms  $\iota^k$  from 2.4(4). To see that the homomorphism

$$\sigma_i: \widetilde{\mathrm{Tor}}_i^R(M, N) \longrightarrow \widetilde{\mathrm{Tor}}_i^R(M, N) \quad \text{given by} \quad [z] \longmapsto ([\partial(z^{\geq k})])_{k \geq 0}$$

provides the desired factorization, note that for  $[z]$  in stable homology  $\widetilde{\mathrm{Tor}}_i^R(M, N)$  one has  $\tau_i \sigma_i([z]) = \tau_i([\partial(z^{\geq k})])_{k \geq 0} = [\partial(z^{\geq 0})] = [\partial(z)] = \delta_i([z])$ .

The comparison map  $\sigma$  is always surjective. Triulzi [20, cor. 6.2.10] proves this in the case of group algebras, and Russell [19, prop. 58] has the general case. As neither thesis is publicly available, we include a proof of surjectivity in Appendix A.

**2.6.** An  $R^\circ$ -module  $M$  is said to have *finite copure flat dimension* if there exists an integer  $n$  such that  $\mathrm{Tor}_i^R(M, E) = 0$  for every injective  $R$ -module  $E$  and all  $i \geq n$ ; see Enochs and Jenda [11]. Prominent examples of such modules are those of finite Gorenstein flat dimension; see [5, thm. 4.14].

**2.7 Proposition.** *For an  $R^\circ$ -module  $M$  of finite copure flat dimension and for all  $i \in \mathbb{Z}$  there are isomorphisms of functors on the category of  $R$ -modules*

$$\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \widetilde{\mathrm{Tor}}_i^R(M, -).$$

*Proof.* By assumption there is an integer  $n$  such that  $\mathrm{Tor}_i^R(M, E) = 0$  for every injective  $R$ -module  $E$  and all  $i \geq n$ . Therefore [4, prop. 2.9] yields  $\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \mathrm{Tor}_i^R(M, -)$  for all  $i \geq n$ . Now [4, 2.3] and Lemma 1.9 finish the proof.  $\square$

**2.8 Remark.** Stable homology  $\widetilde{\mathrm{Tor}}^R(M, -)$  is a homological functor; see [4, (2.4.2)]. Thus, for an  $R^\circ$ -module  $M$  of finite copure flat dimension, the Proposition establishes  $\widetilde{\mathrm{Tor}}^R(M, -)$  as a homological functor. In the proof, the isomorphisms one gets from [4, prop. 2.9] are  $\partial_i$  for  $i \geq n$ , hence the isomorphisms in the statement of Proposition 2.7 are, in fact, the maps  $\sigma_i$  discussed in 2.5; cf. Definition 1.2.

**2.9.** Finitely generated modules over commutative Gorenstein rings have finite Gorenstein flat dimension: this is a result due to Goto; see [5, thm. 1.18, prop. 4.24]. Thus Proposition 2.7 establishes the Main Theorem in the case of Gorenstein rings.

**2.10.** The invariant  $\mathrm{sfi} R$  is the supremum of the flat dimensions of all injective  $R$ -modules. Iwanaga-Gorenstein rings have finite  $\mathrm{sfi}$  by [12, thm. 9.1.7].

Over a ring  $R$  with finite  $\mathrm{sfi}$ , all  $R^\circ$ -modules have finite copure flat dimension. Hence the next result is immediate from Proposition 2.7.

**2.11 Corollary.** *Let  $R$  be a ring with  $\mathrm{sfi} R$  finite. For every  $R^\circ$ -module  $M$  and all  $i \in \mathbb{Z}$ , there are isomorphisms  $\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \mathrm{Tor}_i^R(M, -)$  of functors on the category of  $R$ -modules.*  $\square$

**2.12 Corollary.** *Let  $R$  be a right Noetherian ring and let  $M$  be a finitely generated  $R^\circ$ -module. If  $\mathrm{Ext}_{R^\circ}^i(M, R) = 0$  holds for all  $i \gg 0$ , then for all  $i \in \mathbb{Z}$  there are isomorphisms  $\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \mathrm{Tor}_i^R(M, -)$  of functors on the category of  $R$ -modules.*

*Proof.* Let  $E$  be an injective  $R$ -module. Since  $\mathrm{Ext}_{R^\circ}^i(M, R) = 0$  holds for all  $i \gg 0$ , the isomorphisms  $\mathrm{Tor}_i^R(M, E) \cong \mathrm{Hom}_R(\mathrm{Ext}_{R^\circ}^i(M, R), E)$  from [3, prop. VI.5.3] show that  $M$  has finite copure flat dimension; see 2.6. Thus the claim follows from Proposition 2.7.  $\square$

**Comparison via duality.** With an exact functor  $\mathrm{Hom}(-, E)$  one can, loosely speaking, toggle between  $\mathrm{Ext}$  and  $\mathrm{Tor}$ ; the proof of Corollary 2.12 exemplifies this. To establish the Main Theorem for Artin algebras and commutative complete local rings, we employ a duality  $\mathrm{Hom}(-, E)$  and use that the P-completion of covariant  $\mathrm{Ext}$  agrees with stable cohomology; see Appendix B for details.

**2.13 Theorem.** *Let  $M$  be an  $R^\circ$ -module with a degree-wise finitely generated projective resolution and let  $E$  be an injective  $\mathbb{k}$ -module. For all  $i \in \mathbb{Z}$  there are isomorphisms of functors on the category of  $R^\circ$ -modules*

$$\widetilde{\mathrm{Tor}}_i^R(M, \mathrm{Hom}_{\mathbb{k}}(-, E)) \cong \widetilde{\mathrm{Tor}}_i^R(M, \mathrm{Hom}_{\mathbb{k}}(-, E)).$$

*Proof.* As the P-completion of covariant  $\mathrm{Ext}$  agrees with stable cohomology, [4, thm. A.7] yields isomorphisms,

$$\widetilde{\mathrm{Tor}}_i^R(M, \mathrm{Hom}_{\mathbb{k}}(-, E)) \cong \mathrm{Hom}_{\mathbb{k}}(\widehat{\mathrm{Ext}}_{R^\circ}^i(M, -), E).$$

Fix a degree-wise finitely generated projective resolution  $P \xrightarrow{\sim} M$ . Let  $N$  be a an  $R^\circ$ -module and  $Q \xrightarrow{\sim} N$  be a projective resolution. The quasi-isomorphism  $\mathrm{Hom}_{\mathbb{k}}(N, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{k}}(Q, E)$  is an injective resolution over  $R$ , and  $\mathrm{Hom}_{\mathbb{k}}(Q, E)$

splits as a direct sum  $I \oplus J$ , where  $I$  provides a minimal injective resolution of  $\text{Hom}_{\mathbb{k}}(N, E)$  and  $J$  is acyclic. Now one has:

$$\begin{aligned}
\text{Hom}_{\mathbb{k}}(\widehat{\text{Ext}}_{R^\circ}^i(M, N), E) &\cong \text{Hom}_{\mathbb{k}}(\text{colim}_{k \geq 0} H^i(\text{Hom}_R(P, Q_{\geq k})), E) \\
&\cong \lim_{k \geq 0} \text{Hom}_{\mathbb{k}}(H^i(\text{Hom}_R(P, Q_{\geq k})), E) \\
&\cong \lim_{k \geq 0} H_i(\text{Hom}_{\mathbb{k}}(\text{Hom}_R(P, Q_{\geq k}), E)) \\
&\cong \lim_{k \geq 0} H_i(\text{Hom}_{\mathbb{k}}(\text{Hom}_R(P, \Sigma^k \Omega_k N), E)) \\
&\cong \lim_{k \geq 0} H_i(P \otimes_R \text{Hom}_{\mathbb{k}}(\Sigma^k \Omega_k N, E)) \\
&\cong \lim_{k \geq 0} H_i(P \otimes_R \text{Hom}_{\mathbb{k}}(Q_{\geq k}, E)) \\
&\cong \lim_{k \geq 0} H_i(P \otimes_R (I \oplus J)^{\geq k}) \\
&\cong \widetilde{\text{Tor}}_i^R(M, \text{Hom}_{\mathbb{k}}(N, E)).
\end{aligned}$$

The first isomorphism follows from Lemma B.3. The fourth and the sixth isomorphisms hold since  $Q_{\geq k} \xrightarrow{\sim} \Sigma^k \Omega_k N$  is a projective resolution. The fifth isomorphism holds because  $P$  is degree-wise finitely generated; see [3, prop. VI.5.2]. The last isomorphism follows from 2.4(3) because the complex  $P \otimes_R J$  is acyclic. As the first isomorphism is natural in the  $(N/Q)$  variable, and the subsequent isomorphisms are natural in  $Q$ , it follows that the total isomorphism is natural in  $N$ .  $\square$

The next corollary accounts for the case of Artin algebras in the Main Theorem.

**2.14 Corollary.** *Let  $R$  be an Artin algebra. For every finitely generated  $R^\circ$ -module  $M$  and all  $i \in \mathbb{Z}$ , there are isomorphisms  $\widetilde{\text{Tor}}_i^R(M, -) \cong \text{Tor}_i^R(M, -)$  of functors on the category of finitely generated  $R$ -modules.*

*Proof.* Let  $\mathbb{k}$  be artinian and  $R$  be finitely generated as a  $\mathbb{k}$ -module. Let  $E$  denote the injective hull of  $\mathbb{k}/\text{Jac } \mathbb{k}$  and let  $D(-) = \text{Hom}_{\mathbb{k}}(-, E)$  be the duality functor for  $R$ . By Theorem 2.13 there are isomorphisms of functors on the category of finitely generated  $R$ -modules,

$$\widetilde{\text{Tor}}_i^R(M, -) \cong \widetilde{\text{Tor}}_i^R(M, D(D(-))) \cong \text{Tor}_i^R(M, D(D(-))) \cong \text{Tor}_i^R(M, -). \quad \square$$

**2.15.** Let  $R$  be a commutative local ring and let  $E$  denote the injective hull of its residue field. An  $R$ -module  $N$  is called *Matlis reflexive* if the canonical map  $N \rightarrow \text{Hom}_R(\text{Hom}_R(N, E), E)$  is an isomorphism. For example, finitely generated complete modules, in particular modules of finite length, are Matlis reflexive. With  $D(-) = \text{Hom}_R(-, E)$  the isomorphisms in the proof of Corollary 2.14 remain valid and yield, for a finitely generated  $R$ -module  $M$ , isomorphisms

$$\widetilde{\text{Tor}}_i^R(M, -) \cong \text{Tor}_i^R(M, -)$$

of functors on the subcategory of Matlis reflexive  $R$ -modules.

Over a commutative complete local ring, Matlis duality establishes that all finitely generated modules and all artinian modules are Matlis reflexive. The next corollary accounts for the case of complete local rings in the Main Theorem.



**2.16 Corollary.** *Let  $R$  be a commutative noetherian complete local ring. For all finitely generated  $R$ -modules  $M$  and all  $i \in \mathbb{Z}$ , there are isomorphisms of functors on the category of finitely generated  $R$ -modules  $\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \widehat{\mathrm{Tor}}_i^R(M, -)$ .  $\square$*

**2.17 Remark.** For non-negative integers  $i$ , the isomorphisms in Corollaries 2.12 and 2.14 were proved by Yoshino [22, thms. 5.3 and 5.5]. Indeed, Yoshino introduces and studies in *loc. cit.* the *Tate–Vogel completion* of covariant half exact functors. For such a functor  $T$ —our interest is in  $\mathrm{Tor}_i^R(M, -)$  for an  $R^\circ$ -module  $M$ —the Tate–Vogel completion  $T^\wedge$  is given by  $T^\wedge(N) = \lim_{k \geq 0} S^k S_k T(N)$ , for every  $R$ -module  $N$ . Here  $S_k T$  denotes the  $k^{\mathrm{th}}$  *left satellite* of the functor  $T$ ; see [3, chap. III].

For  $i \in \mathbb{Z}$  set  $T_i = \mathrm{Tor}_i^R(M, -)$ . As  $T_i$  is 0 for  $i < 0$ , Yoshino only considers the Tate–Vogel completion  $T_i^\wedge$  for  $i \geq 0$ . However, for  $i \geq 0$  dimension shifting yields  $S_k T_i \cong T_{k+i}$ . Thus, in the definition of the Tate–Vogel completion of  $T_i$  we may replace  $S_k T_i$  by  $T_{k+i}$ , so that in the limit one gets a Tate–Vogel completion  $T_i^\wedge$  of  $T_i$  for all  $i \in \mathbb{Z}$ . By 1.5 and Definition 2.1 this is complete homology  $\widehat{\mathrm{Tor}}_i^R(M, -)$ .

**2.18 Remark.** In his thesis [19], Russell studies an *asymptotic stabilization* of the tensor product and compares it to Yoshino’s work [22] discussed above, to stable homology (which he calls Vogel homology), and to complete homology (which, not having access to [20], he refers to as “mirror-Mislin”). As already mentioned, Russel [19, prop. 58] proves surjectivity of the comparison map from stable to complete homology; furthermore [19, prop. 57] coincides with Corollary 2.12.

### 3. COMPARISON TO TATE HOMOLOGY

Tate (co)homology was originally defined for modules over finite group algebras. Through works of Iacob [14] and Veliche [21] the theories have been generalized to the extent that one can talk about Tate homology  $\widehat{\mathrm{Tor}}^R(M, N)$  and Tate cohomology  $\widehat{\mathrm{Ext}}_R(M, N)$  for modules over any ring, provided that the first argument,  $M$ , has a complete projective resolution; see 3.1.

Stable cohomology and the P-completion of covariant  $\mathrm{Ext}$  always agree, and they coincide with Tate cohomology whenever the latter is defined; see Appendix B. The questions of when stable homology and complete homology (the J-completion of  $\mathrm{Tor}$ ) agree, and when they coincide with Tate homology, do not yet have satisfactory answers. In [4] we prove that stable homology,  $\widetilde{\mathrm{Tor}}$ , agrees with Tate homology over a ring  $R$  if and only if every Gorenstein projective  $R^\circ$ -module is Gorenstein flat; see 3.1 and 3.4. In this section we prove that complete homology  $\widehat{\mathrm{Tor}}$  agrees with Tate homology under the exact same condition. We give two proofs of this fact; the first one uses the result for stable homology, and the second is independent thereof.

**3.1.** An acyclic complex  $T$  of projective  $R^\circ$ -modules is called *totally acyclic* if  $\mathrm{Hom}_{R^\circ}(T, P)$  is acyclic for every projective  $R^\circ$ -module  $P$ . An  $R^\circ$ -module  $G$  is called *Gorenstein projective* if there exists a totally acyclic complex  $T$  of projective  $R^\circ$ -modules with  $\mathrm{Coker}(T_1 \rightarrow T_0) \cong G$ . A *complete projective resolution* of an  $R^\circ$ -module  $M$  is a diagram  $T \xrightarrow{\varpi} P \xrightarrow{\simeq} M$ , where  $T$  is a totally acyclic complex of projective  $R^\circ$ -modules,  $P \xrightarrow{\simeq} M$  is a projective resolution, and  $\varpi_i$  is an isomorphism for  $i \gg 0$ ; see [21, sec. 2]. A module has a complete projective resolution if and only if it has finite Gorenstein projective dimension; see [21, thm. 3.4].

Let  $M$  be an  $R^\circ$ -module with a complete projective resolution  $T \rightarrow P \rightarrow M$ , and let  $N$  be an  $R$ -module. The *Tate homology* of  $M$  and  $N$  over  $R$  is the collection of  $\mathbb{k}$ -modules  $\widehat{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N)$  for  $i \in \mathbb{Z}$ ; see [14, sec. 2].

If  $R$  is noetherian and  $M$  is a finitely generated  $R^\circ$ -module that has a complete projective resolution, then one has  $\widehat{\mathrm{Tor}}^R(M, -) \cong \widehat{\mathrm{Tor}}^R(M, -)$ ; see [4, thm. 6.4]. Next comes the analogous statement for complete homology.

**3.2 Proposition.** *Let  $R$  be a noetherian ring and let  $M$  be a finitely generated  $R^\circ$ -module that has a complete projective resolution. For all  $i \in \mathbb{Z}$ , there are isomorphisms of functors on the category of  $R$ -modules,*

$$\widehat{\mathrm{Tor}}_i^R(M, -) \cong \widehat{\mathrm{Tor}}_i^R(M, -).$$

*Proof.* It follows from [21, 2.4.1, 3.4] that  $\mathrm{Ext}_{R^\circ}^i(M, R) = 0$  holds for all  $i \gg 0$ . Therefore, by Corollary 2.12, there are isomorphisms  $\widehat{\mathrm{Tor}}_i^R(M, -) \cong \widehat{\mathrm{Tor}}_i^R(M, -)$  of functors for all  $i \in \mathbb{Z}$ . Now [4, thm. 6.4] completes the proof.  $\square$

With the next observation we can give an alternate proof of Proposition 3.2, one that does not rely upon knowing how stable homology compares to Tate homology.

**3.3.** Let  $M$  be an  $R^\circ$ -module with a complete projective resolution  $T \rightarrow P \rightarrow M$ . The morphism  $T \rightarrow P$  induces a morphism  $v$  of homological functors, whose components  $v_i: \widehat{\mathrm{Tor}}_i^R(M, -) \rightarrow \mathrm{Tor}_i^R(M, -)$  are isomorphisms for all  $i \gg 0$ . To prove that Tate homology  $\widehat{\mathrm{Tor}}^R(M, -)$  agrees with complete homology  $\widehat{\mathrm{Tor}}^R(M, -)$ , it suffices by Lemma 1.9 to show that  $\widehat{\mathrm{Tor}}^R(M, E)$  vanishes for all injective  $R$ -modules  $E$ .

*Alternate proof of 3.2.* It follows from [21, 2.4.1] and [1, thm. 3.1] that  $M$  has a complete projective resolution  $T \rightarrow P \rightarrow M$  with  $T$  and  $P$  degree-wise finitely generated. Let  $E$  be an injective  $R$ -module; the complex

$$T \otimes_R E \cong T \otimes_R \mathrm{Hom}_R(R, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_{R^\circ}(T, R), E)$$

is acyclic; the last isomorphism is [3, prop. VI.5.2]. Thus one has  $\widehat{\mathrm{Tor}}_i^R(M, E) = \mathrm{H}_i(T \otimes_R E) = 0$  for all  $i \in \mathbb{Z}$ , and now 3.3 finishes the argument.  $\square$

**3.4.** An  $R^\circ$ -module  $G$  is called *Gorenstein flat* if there exists an acyclic complex  $T$  of flat  $R^\circ$ -modules with  $\mathrm{Coker}(T_1 \rightarrow T_0) \cong G$  and such that  $T \otimes_R E$  is acyclic for every injective  $R$ -module  $E$ . In general, it is not known whether or not Gorenstein projective modules are Gorenstein flat; see also Emmanouil [9, thm. 2.2].

We show in [4, thm. 6.7] that one has  $\widehat{\mathrm{Tor}}^R(M, -) \cong \widehat{\mathrm{Tor}}^R(M, -)$  for every  $R^\circ$ -module  $M$  with a complete projective resolution if and only if every Gorenstein projective  $R^\circ$ -module is Gorenstein flat. Here is the result for complete homology.

**3.5 Proposition.** *The following conditions on  $R$  are equivalent.*

- (i) *Every Gorenstein projective  $R^\circ$ -module is Gorenstein flat.*
- (ii) *For every  $R^\circ$ -module  $M$  that has a complete projective resolution and for all  $i \in \mathbb{Z}$  there are isomorphisms of functors on the category of  $R$ -modules,*

$$\widehat{\mathrm{Tor}}_i^R(M, -) \cong \widehat{\mathrm{Tor}}_i^R(M, -).$$

*Proof.* (i)  $\implies$  (ii): Let  $M$  be an  $R^\circ$ -module that has a complete projective resolution. By assumption,  $M$  has finite Gorenstein flat dimension, so there is an  $n \geq 0$  such that  $\mathrm{Tor}_i^R(M, E) = 0$  holds for every injective  $R$ -module  $E$  and all  $i > n$ ; see [5, thm. 4.14]. The desired isomorphisms of functors now follow from Proposition 2.7 and [4, thm. 6.7].

(ii)  $\implies$  (i): Let  $M$  be a Gorenstein projective  $R^\circ$ -module and let  $T$  be a totally acyclic complex of projective  $R^\circ$ -modules with  $M \cong \mathrm{Coker}(T_1 \rightarrow T_0)$ . By assumption there are isomorphisms

$$H_i(T \otimes_R E) \cong \widetilde{\mathrm{Tor}}_i^R(M, E) = 0$$

for every injective  $R$ -module  $E$  and all  $i \in \mathbb{Z}$ ; see 2.2 and 3.1. Thus  $T \otimes_R E$  is acyclic for every injective  $R$ -module  $E$ , and hence  $M$  is Gorenstein flat.  $\square$

Again, there is an alternate proof that does not require any knowledge of how stable homology compares to Tate homology.

*Alternate proof of 3.5.* In the proof above, only the implication (i)  $\implies$  (ii) references stable homology. Let  $M$  be an  $R^\circ$ -module that has a complete projective resolution  $T \rightarrow P \rightarrow M$ . Assuming that every Gorenstein projective  $R^\circ$ -module is Gorenstein flat, it follows that the complex  $T \otimes_R E$  is acyclic for every injective  $R$ -module  $E$ ; see [9, thm. 2.2]. Thus one has  $\widetilde{\mathrm{Tor}}_i^R(M, E) = H_i(T \otimes_R E) = 0$  for all  $i \in \mathbb{Z}$ , and 3.3 finishes the argument.  $\square$

**3.6 Remark.** The similarity of Proposition 3.5 to [4, thm. 6.7] means that for a ring  $R$  the following conditions are equivalent:

- (i) For every  $R^\circ$ -module  $M$  with a complete projective resolution there are isomorphisms of functors  $\widetilde{\mathrm{Tor}}_i^R(M, -) \cong \widehat{\mathrm{Tor}}_i^R(M, -)$  for all  $i \in \mathbb{Z}$ .
- (ii) For every  $R^\circ$ -module  $M$  with a complete projective resolution there are isomorphisms of functors  $\widehat{\mathrm{Tor}}_i^R(M, -) \cong \widetilde{\mathrm{Tor}}_i^R(M, -)$  for all  $i \in \mathbb{Z}$ .

As a contrast to 3.6, we make two remarks to clarify that stable homology agrees with complete homology in cases where Tate homology is not defined.

**3.7 Remark.** Over a commutative artinian ring  $R$  that is not Gorenstein, there exist finitely generated modules that do not have a complete projective resolution. For example, let  $\mathfrak{m}$  be a maximal ideal of  $R$  such that the local ring  $R_{\mathfrak{m}}$  is not Gorenstein, then  $R/\mathfrak{m}$  is such a module; see [5, prop. 2.17 and thm. 2.19]. In this case, Tate homology “ $\widehat{\mathrm{Tor}}^R(R/\mathfrak{m}, -)$ ” is not defined, but by the Main Theorem, complete homology  $\widetilde{\mathrm{Tor}}^R(R/\mathfrak{m}, N)$  still agrees with stable homology  $\widehat{\mathrm{Tor}}^R(R/\mathfrak{m}, N)$  for all finitely generated  $R$ -modules  $N$ .

The same phenomenon can occur over a ring with finite sfli; cf. Corollary 2.11.

**3.8 Remark.** Let  $R$  be von Neumann regular, then every  $R^\circ$ -module is flat. It follows that stable homology  $\widetilde{\mathrm{Tor}}^R(M, -)$  and complete homology  $\widehat{\mathrm{Tor}}^R(M, -)$  agree and vanish for every  $R^\circ$ -module  $M$ ; see 2.2 and Corollary 2.11. It is, however, possible that Tate homology “ $\widehat{\mathrm{Tor}}^R(M, -)$ ” is not defined. To see this, let  $R$  be a commutative von Neumann regular ring over which there are modules of infinite projective dimension; the existence of such rings is proved by Osofsky [18, 3.1]. Let  $G$  be an  $R$ -module with a complete projective resolution  $T \rightarrow P \rightarrow G$ . It

follows from [6, prop. 7.6] that  $T$  is contractible. Thus,  $G$  has a projective syzygy and hence it has finite projective dimension. An  $R$ -module  $M$  of infinite projective dimension, therefore, does not have a complete projective resolution, and Tate homology “ $\widehat{\text{Tor}}^R(M, -)$ ” is not defined.

Over a ring  $R$  with finite sfi  $R$ , every Gorenstein projective  $R^\circ$ -module is Gorenstein flat; cf. 3.4. The situation is the same if  $R$  is commutative and artinian, and more generally if  $R$  is right-perfect: Let  $T$  be a totally acyclic complex of projective  $R^\circ$ -modules, let  $I$  be an injective  $R$ -module and  $E$  be a faithfully injective  $\mathbb{k}$ -module. The complex  $T \otimes_R I$  is acyclic if and only if the dual complex  $\text{Hom}_{\mathbb{k}}(T \otimes_R I, E) \cong \text{Hom}_{R^\circ}(T, \text{Hom}_{\mathbb{k}}(I, E))$  is acyclic. The  $R^\circ$ -module  $\text{Hom}_{\mathbb{k}}(I, E)$  is flat and, by the assumption on  $R$ , that means projective. By 3.1 and the isomorphism above, the complex  $\text{Hom}_{\mathbb{k}}(T \otimes_R I, E)$  is acyclic, whence  $T \otimes_R I$  is acyclic.

The general question of whether or not Gorenstein projective modules are Gorenstein flat remains open. We have no example of a ring over which complete homology agrees with stable homology, and over which Gorenstein projective modules are not known to be Gorenstein flat.

#### APPENDIX A

Let  $M$  be an  $R^\circ$ -module and let  $N$  be an  $R$ -module. We show that the natural comparison maps  $\sigma_i: \widehat{\text{Tor}}_i^R(M, N) \rightarrow \text{Tor}_i^R(M, N)$  described in 2.5 are surjective.

**A.1.** Let  $P \xrightarrow{\sim} M$  be a projective resolution and  $N \xrightarrow{\sim} I$  be an injective resolution. The complex  $P \otimes_R I$  is the direct sum totalization—and  $P \overline{\otimes}_R I$  is the product totalization—of the double complex  $D = (D_m^n)_{m,n \geq 0}$  with  $D_m^n = P_m \otimes_R I^n$ , anti-commuting squares, and exact rows:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P_m \otimes I^0 & \longrightarrow & P_m \otimes I^1 & \longrightarrow & P_m \otimes I^2 & \longrightarrow & \cdots \longrightarrow P_m \otimes I^n \longrightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P_1 \otimes I^0 & \longrightarrow & P_1 \otimes I^1 & \longrightarrow & P_1 \otimes I^2 & \longrightarrow & \cdots \longrightarrow P_1 \otimes I^n \longrightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P_0 \otimes I^0 & \longrightarrow & P_0 \otimes I^1 & \longrightarrow & P_0 \otimes I^2 & \longrightarrow & \cdots \longrightarrow P_0 \otimes I^n \longrightarrow \cdots
 \end{array}$$

For every  $k > 0$  the complex  $P \otimes_R I^{\geq k}$  is the direct sum totalization of the double complex obtained by removing the columns  $0, \dots, k-1$  from  $D$ .

Let  $k \geq 0$ ; an element in  $(P \otimes_R I^{\geq k})_i$  is a sequence  $v = (v^n)_{n \geq k}$  with  $v^n$  in  $D_{i+n}^n$ , and  $v^n = 0$  for  $n \gg 0$ . Notice that one has  $v^n = 0$  for  $n < -i$ . That is,  $v$  is supported at  $D_{i+n}^n$  for finitely many  $n \geq \max\{k, -i\}$ .

We make repeated use of the technique of compressing cycles in tensor products. Though it is standard, we develop it here in the notation of the double complex  $D$ .

**A.2.** Fix  $i \in \mathbb{Z}$  and  $k \geq 0$ . Let  $v$  be an element in  $(P \otimes_R I^{\geq k})_i$  supported at  $D_{i+n}^n$  for  $h \leq n \leq j$ ; its boundary  $\partial(v)$  in  $(P \otimes_R I^{\geq k})_{i-1}$  is then supported at  $D_{i-1+n}^n$  for  $h \leq n \leq j+1$ , and the following hold:

- (a) If  $\partial(v)$  is supported at  $D_{i-1+n}^n$  for  $h \leq n \leq j'$  with  $h < j' \leq j$ , then there exists an element  $u$  in  $(P \otimes_R I^{\geq k})_{i+1}$  supported at  $D_{i+1+n}^n$  for  $j' - 1 \leq n \leq j - 1$ , such that  $v - \partial(u)$  is supported at  $D_{i+n}^n$  for  $h \leq n \leq j' - 1$ .
- (b) If  $\partial(v) = 0$ , then for every  $e$  with  $\max\{k, -i\} \leq e \leq h$  here exists an element  $u_e$  in  $(P \otimes_R I^{\geq k})_{i+1}$  supported at  $D_{i+1+n}^n$  for  $e \leq n \leq j - 1$ , such that  $v - \partial(u_e)$  is supported at  $D_{i+e}^e$ .

Indeed, start at the last component of  $v$ , that is, at  $v^j \in D_{i+j}^j$ . That element is a cycle in row  $i + j$  by the assumption  $j' \leq j$ , so there is an element  $u^{j-1}$  in  $D_{i+j}^{j-1}$  with  $\partial^h(u^{j-1}) = v^j$ . Thus  $v' = v - \partial(u^{j-1})$  is supported at  $D_{i+n}^n$  for  $h \leq n \leq j - 1$ . If  $j' = j$ , then  $u^{j-1}$  is the element claimed in (a). If not, notice that one has  $\partial(v') = \partial(v)$  and repeat; after  $j - j' + 1$  iterations one has the desired element  $u$ .

Now, if  $v$  is a cycle, then  $\partial(v)$  has empty support. Thus the procedure above can be repeated until  $v - \partial(u)$  is supported at  $D_{i+h}^h$ . From there it can be repeated to produce a  $u$  with  $v - \partial(u)$  supported at  $D_{i+e}^e$  as long as  $D_{i+e}^e$  is within the boundaries of the (truncated) double complex.

We can now describe the elements in  $\widetilde{\text{Tor}}_i^R(M, N) \cong \lim_{k \geq 0} \text{H}_i(P \otimes_R I^{\geq k})$  in such a way that surjectivity of the comparison map becomes almost trivial.

**A.3.** Fix  $i \in \mathbb{Z}$  and set  $d = \max\{0, -i\}$ ; one has

$$\lim_{k \geq 0} \text{H}_i(P \otimes_R I^{\geq k}) \cong \lim_{k \geq d} \text{H}_i(P \otimes_R I^{\geq k}).$$

An element in  $\lim_{k \geq d} \text{H}_i(P \otimes_R I^{\geq k})$  is a compatible family of homology classes  $([w^{\geq k}])_{k \geq d}$ , where  $w^{\geq k}$  is a cycle of degree  $i$  in  $P \otimes_R I^{\geq k}$ ; by A.2(b) we may assume that  $w^{\geq k}$  is supported at the left edge of the truncated double complex, i.e. at  $D_{i+k}^k$ . Compatibility means that for each  $k \geq d$  the cycle  $w^{\geq k} - w^{\geq k+1}$  in  $P \otimes_R I^{\geq k}$  is a boundary:  $w^{\geq k} - w^{\geq k+1} = \partial(x)$  for some  $x$  in  $P \otimes_R I^{\geq k}$ . It is supported at  $D_{i+k}^k$  and  $D_{i+1+k}^{k+1}$ , so by A.2(a) there exists an element  $v^k \in D_{i+1+k}^k$  with  $\partial(v^k) = w^{\geq k} - w^{\geq k+1}$ . Observe that one has  $\partial^v(v^k) = w^{\geq k}$  and  $\partial^h(v^k) = -w^{\geq k+1}$ .

To see that  $\sigma_i$  is surjective, let an element  $([w^{\geq k}])_{k \geq d}$  in  $\lim_{k \geq d} \text{H}_i(P \otimes_R I^{\geq k})$  be given. Choose for each  $k \geq d$  an element  $v^k$  as above. The family  $v = (v^k)_{k \geq d}$  is an element in  $(P \otimes_R I)_{i+1}$  and  $\partial(v) = w^{\geq d}$  is in  $P \otimes_R I$ ; the equivalence class  $[v]$  in  $(P \otimes_R I)_{i+1}$  is a cycle with  $[\partial(v^{\geq k})] = [w^{\geq k}]$ . Thus one has  $\sigma_i([v]) = ([w^{\geq k}])_{k \geq d}$ .

Next we briefly discuss injectivity.

**A.4.** Adopt the notation from A.3. Consider a cycle in  $(P \otimes_R I)_{i+1}$ , represented by  $z = (z^k)_{k \geq d}$  in  $(P \otimes_R I)_{i+1}$ , and assume  $\sigma_i([z]) = 0$ . To prove that  $[z]$  is zero in  $\text{H}_{i+1}(P \otimes_R I) = \widetilde{\text{Tor}}_i^R(M, N)$  amounts to proving the existence of an element  $v$  in  $(P \otimes_R I)_{i+2}$  such that  $z - \partial(v)$  has finite support, i.e. belongs to  $(P \otimes_R I)_{i+1}$ .

Consider  $\sigma_i([z]) = ([\partial(z^{\geq k})])_{k \geq d}$  in  $\lim_{k \geq d} \text{H}_i(P \otimes_R I^{\geq k})$ . As  $\partial(z)$  is in  $(P \otimes_R I)_i$ , it follows that  $\partial(z^{\geq k})$  for  $k \gg 0$  is supported at only one component of the double complex. That is, one can choose  $k' > d$  such that for all  $k \geq k'$  the element  $b^k = \partial(z^{\geq k})$  in  $(P \otimes_R I^{\geq k})_i$  is supported at  $D_{i+k}^k$ ; in fact, it is  $\partial^v(z^k) = -\partial^h(z^{k-1})$ .

The assumption  $\sigma_i([z]) = 0$  implies, in particular, that for every  $k \geq k'$  there exists an element  $x^k$  in  $(P \otimes_R I^{\geq k})_{i+1}$  with  $\partial(x^k) = b^k$ . By A.2(a) we may assume that  $x^k$  is supported at  $D_{i+k+1}^k$  where also  $z^k$  resides. Thus one has  $\partial^v(x^k) = b^k = \partial^v(z^k)$  and  $\partial^h(x^k) = 0$ .

We have not been able to verify that the comparison map is injective, i.e. an isomorphism, in situations that are not covered by Proposition 2.7. In the primitive case of that result,  $M$  is a module with  $\text{Tor}_{>0}^R(M, E) = 0$  for all injective  $R$ -modules  $E$ , so the columns of  $D$  are exact, and a straightforward diagram chase then produces the element  $v$  sought after in A.4. As the index  $i$  and the cycle  $z$  are arbitrary, this shows that the comparison map is injective. However, we already knew that and more from Proposition 2.7, so to conclude we ask:

**A.5 Question.** Let  $M$  be an  $R^\circ$ -module and  $i$  be an integer. If for every  $R$ -module  $N$  there is an isomorphism  $\widetilde{\text{Tor}}_i^R(M, N) \cong \widetilde{\text{Tor}}_i^R(M, N)$ , is then the comparison map  $\sigma_i$  an isomorphism?

## APPENDIX B

Kropholler states in [15, sec. 3.3] that the three generalizations of Tate cohomology due to Benson and Carlson [2], Mislin [16], and P. Vogel [13] are isomorphic. A proof of this claim can be engineered by mimicking arguments in Nucinkis's paper [17]. The purpose of this appendix is to sketch how this would work.

First, a word on notation.

**B.1 Notation.** For  $R$ -modules  $M$  and  $N$  we use, like in Definition 2.1, the symbols  $\widehat{\text{Ext}}_R^i(M, N)$  for  $i \in \mathbb{Z}$  to denote the P-completion of the cohomological functor  $\text{Ext}_R^i(M, -)$  evaluated at  $N$ . This is the standard notation for Tate cohomology, but no ambiguity arises as complete and Tate cohomology agree whenever the latter is defined; see 3.1 and Cornick and Kropholler [8, thm. 1.2].

**B.2.** With extra notation to distinguish them, the three families of cohomology modules are defined as follows,

$$\begin{aligned} \text{(complete, Mislin)} \quad & \widehat{\text{Ext}}_{\text{comp.}}^i(M, N) = \text{colim}_{k \geq 0} S_k \text{Ext}_R^{k+i}(M, N), \\ \text{(stable, Vogel)} \quad & \widehat{\text{Ext}}_{\text{stable}}^i(M, N) = H_{-i}(\text{Hom}_R(P, Q) / \overline{\text{Hom}}(P, Q)), \text{ and} \\ \text{(Benson and Carlson)} \quad & \widehat{\text{Ext}}_{\text{B\&C}}^i(M, N) = \text{colim}_{k \geq 0} \underline{\text{Hom}}(\Omega_k M, \Omega_{k-i} N). \end{aligned}$$

In the first line,  $S_k$  denotes the  $k^{\text{th}}$  left satellite; see [3, chap. III]. In the second line,  $P \xrightarrow{\simeq} M$  and  $Q \xrightarrow{\simeq} N$  are projective resolutions, and  $\overline{\text{Hom}}(P, Q)$  is the subcomplex of  $\text{Hom}_R(P, Q)$  with modules  $\overline{\text{Hom}}(P, Q)_i = \coprod_{n \in \mathbb{Z}} \text{Hom}_R(P_n, Q_{n+i})$ ; here the coproduct replaces the product used for the Hom functor. In the third line,  $\underline{\text{Hom}}$  is notation for stable Hom: the  $\mathbb{k}$ -module of homomorphisms modulo those that factor through a projective  $R$ -module. Finally,  $\Omega_n M$  is the  $n^{\text{th}}$  syzygy,  $\text{Coker}(P_{n+1} \rightarrow P_n)$ , in a projective resolution,  $P \xrightarrow{\simeq} M$ ; by Schanuel's lemma it is unique up to a projective summand.

Mislin [16, sec. 4] proved that his complete cohomology theory is isomorphic to Benson and Carlson's  $\widehat{\text{Ext}}_{\text{B\&C}}$ . Thus it is sufficient to show that stable cohomology is

isomorphic to  $\widehat{\text{Ext}}_{\text{B\&C}}$ . Our goal is to construct a morphism

$$\widehat{\text{Ext}}_{\text{stable}}(M, -) \xrightarrow{\mu} \widehat{\text{Ext}}_{\text{B\&C}}(M, -)$$

of cohomological functors, such that the maps  $\mu_i^N: \widehat{\text{Ext}}_{\text{stable}}^i(M, N) \rightarrow \widehat{\text{Ext}}_{\text{B\&C}}^i(M, N)$  are isomorphisms and compatible with the connecting homomorphisms.

**Construction of  $\mu$ .** Fix an  $i \in \mathbb{Z}$  and an  $R$ -module  $N$ . An element

$$\widehat{\varphi} \in \widehat{\text{Ext}}_{\text{stable}}^i(M, N) = H_{-i}(\text{Hom}_R(P, Q)/\overline{\text{Hom}}(P, Q))$$

is represented by a homomorphism  $\varphi$  of homological degree  $-i$ , which is a chain map in high degrees, i.e., for  $k \gg 0$  the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_{k+1} & \longrightarrow & P_k & \longrightarrow & \Omega_k M \\ & & \downarrow \varphi_{k+1} & & \downarrow \varphi_k & & \downarrow \tilde{\varphi}_k \\ \cdots & \longrightarrow & Q_{k+1-i} & \longrightarrow & Q_{k-i} & \longrightarrow & \Omega_{k-i} N \end{array}$$

is commutative up to a sign  $(-1)^i$ . The right-hand square defines  $\tilde{\varphi}_k$ . In this way,  $\varphi$  defines element  $\tilde{\varphi}$  in  $\widehat{\text{Ext}}_{\text{B\&C}}^i(M, N)$ . To see that this yields a homomorphism

$$\mu_i^N: \widehat{\text{Ext}}_{\text{stable}}^i(M, N) \longrightarrow \widehat{\text{Ext}}_{\text{B\&C}}^i(M, N),$$

it must be verified that  $\tilde{\varphi}$  is independent of the choice of representative  $\varphi$  of  $\widehat{\varphi}$  in  $\widehat{\text{Ext}}_{\text{stable}}^i(M, N)$ . If  $\widehat{\varphi} = \widehat{\psi}$  in  $\widehat{\text{Ext}}_{\text{stable}}^i(M, N)$ , then  $\varphi - \psi$  is 0-homotopic in high degrees, so for every  $k \gg 0$  the induced homomorphism  $\tilde{\varphi}_k - \tilde{\psi}_k: \Omega_k M \rightarrow \Omega_{k-i} N$  factors through the projective module  $Q_{k-i}$ , whence it is zero in  $\underline{\text{Hom}}(\Omega_k M, \Omega_{k-i} N)$ .

It is straightforward to verify that  $\mu_i^N$  as defined above is natural in  $N$  and that the family  $\mu^N$  is compatible with the connecting homomorphisms.

**Injectivity.** Let  $\widehat{\varphi}$  be an element in the kernel of  $\mu_i^N$ . For every  $k \gg 0$  the induced morphism  $\tilde{\varphi}_k: \Omega_k M \rightarrow \Omega_{k-i} N$  factors through a projective  $R$ -module  $L$  and further through the surjection  $Q_{k-i} \rightarrow \Omega_{k-i} N$ ,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_{k+1} & \longrightarrow & P_k & \longrightarrow & \Omega_k M \\ & & \downarrow \varphi_{k+1} & \searrow \sigma_k & \downarrow \varphi_k & \nearrow \sigma_{k-1} & \downarrow \tilde{\varphi}_k \\ & & & & & L & \\ & & & & & \swarrow & \\ \cdots & \longrightarrow & Q_{k+1-i} & \longrightarrow & Q_{k-i} & \longrightarrow & \Omega_{k-i} N \end{array}$$

It is now straightforward to construct the homotopies  $\sigma_n: P_n \rightarrow Q_{n+1-i}$  for  $n \geq k$ , so  $\varphi$  is 0-homotopic in high degrees, i.e. one has  $\widehat{\varphi} = 0$  in  $\widehat{\text{Ext}}_{\text{stable}}^i(M, N)$ .

**Surjectivity.** An element  $\tilde{\varphi}$  in  $\widehat{\text{Ext}}_{\text{B\&C}}^i(M, N)$  is a family of elements in the direct system of modules  $\underline{\text{Hom}}(\Omega_k M, \Omega_{k-i} N)$ . Such a family is determined by an element  $\underline{\varphi}$  in  $\underline{\text{Hom}}(\Omega_k M, \Omega_{k-i} N)$  for some  $k \gg 0$ , and  $\underline{\varphi}$  is represented by a homomorphism  $\varphi \in \text{Hom}_R(\Omega_k M, \Omega_{k-i} N)$ . Lifting  $\varphi$  to a morphism of projective resolutions  $P_{\geq k} \rightarrow \Sigma^i Q_{\geq k-i}$  yields an element  $\widehat{\varphi}$  in  $\widehat{\text{Ext}}_{\text{stable}}^i(M, N)$  with  $\mu_i^N(\widehat{\varphi}) = \tilde{\varphi}$ .

The next result is used in the proof of Theorem 2.13. It is dual to 2.4(3) and has a similar proof, which can also be extracted from [16, sec. 4].

**B.3 Lemma.** *Let  $M$  and  $N$  be  $R$ -modules with projective resolutions  $P \xrightarrow{\sim} M$  and  $Q \xrightarrow{\sim} N$ . For every  $i \in \mathbb{Z}$  there is an isomorphism of  $\mathbb{k}$ -modules*

$$\widehat{\mathrm{Ext}}_R^i(M, N) \cong \varinjlim_{k \geq 0} H^i(\mathrm{Hom}_R(P, Q_{\geq k}))$$

which is natural in the second argument.

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